Comprehensive Exam in Advanced Algorithms : Spring 2004

1. Describe the growth of each of the following functions as either *polynomial, superpolynomial,* or *exponential.* A function $f(n)$ is said to be polynomial if there exists some positive constant k such that $f(n) = O(n^k)$. A function $f(n)$ is said to be *superpolynomial* if for every positive constant k, $f(n) = \omega(n^k)$ and for every positive constant c, $f(n) = o(c^n)$. A function $f(n)$ is said to be *exponential* if there exists some positive constant k such that $f(n) = \Omega(k^n)$. Justify all answers with a short mathematical explanation.

   (a) $f(n) = (2 - \frac{1}{n})^n$

   (b) $f(n) = n^{\frac{1}{\sqrt{n}}}$

   (c) $f(n) = 2^{\frac{(\log n)^2}{\log \log n}}$

   (d) $f(n) = \sum_{i=1}^{\lceil \log n \rceil} 2^i$

(e) (extra credit) $\log^*(n)$ is defined to be the number of logarithms one can successively take of the number n until you get something less than or equal to 1. For example, if $n = 16$, then

$$\log 16 = 4$$

$$\log\log 16 = 2$$

$$\log\log\log 16 = 1$$

So $\log^*(16) = 3$ because we could take 3 successive logarithms before the result turns out less than or equal to 1 (actually equal to 1 in this case). Determine which function is asymptotically larger: $\log(\log^*(n))$ or $\log^*(\log n)$. Justify your answer. (HINT: Consider $\log^*(\log n)$. You've already taken a log... Think about what that expression is in terms of $\log^*(n)$.)

2. An **independent set** of a graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V'. The Independent Set Decision Problem is the following: Given a graph $G = (V, E)$ and a positive integer K, does there exist an independent set in the graph G of size greater than or equal to K?

   A **clique** in a graph $G = (V, E)$ is a complete subgraph of G. In other words (slightly more formally), a clique is a subset $V' \subseteq V$ such that for every two vertices $u, v \in V'$, the edge $(u, v) \in E$. The Clique Decision Problem is the following: Given a graph $G = (V, E)$ and a positive integer K, does there exist a clique in the graph G of size greater than or equal to K? The Clique Decision Problem is NP-complete. Show that the Independent Set Decision Problem is NP-complete as well. (HINT: Consider the dual of the graph G...)

3. Assume that we Huffman encode the following message (the one with the letters). Using the same Huffman tree as you got for the encoding, decode the message in binary beneath the original message. Recall that 0 means "left" and 1 means "right."

$$PSSSSSYYYYYYOOOOOONNN$$

01001101000011

4. A **polygon** is a piecewise-linear, closed curve in the plane. A **triangulation** of a polygon is a set T of chords of the polygon that divide the polygon into disjoint triangles. We wish to find the optimal triangulation of the following graph G represented here as an adjacency matrix M.

$$M = \begin{matrix} 0 & 15 & 3 & 47 & 2 \\ 31 & 0 & 21 & 23 & 11 \\ 33 & 9 & 0 & 12 & 7 \\ 52 & 25 & 73 & 0 & 44 \\ 1 & 6 & 4 & 8 & 0 \end{matrix}$$

We think of the vertices of this adjacency matrix as being numbered from 1 to 5 (from left to right). The five vertices are located in the plane such that v_1 through v_5 appear in clockwise order. For $2 \leq i \leq j \leq 5$, define $t[i,j]$ to be the weight of the optimal polygon triangulation of the polygon formed by vertices v_i through v_j. Then the recursive formula for $t[i,j]$ is as follows.

$$t[i,j] = \begin{matrix} 0 & \text{if } i = j \\ \min_{i \leq k \leq j-1}\{t[i,k] + t[k+1,j] + w(\Delta v_{i-1} v_k v_j)\} & \text{if } i < j \end{matrix}$$

where $w(\Delta v_{i-1} v_k v_j) = M_{i-1,k} + M_{k,j} + M_{j,i-1}$. ($M_{i-1,k}$ represents the element in row $i-1$, column k of the adjacency matrix, etc.) Use the method of **dynamic programming** to determine an optimal polygon triangulation for the given graph G.

SCRATCH PAPER